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PSFs and Sampling of Images in Astronomy

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1 Moffat PSFs

[Moffat](#) proposed using a softened exponential profile to model a PSF: determines the overall shape of the PSF, while α is a scale factor.

$$I(r) = I_0 \left[1 + \left(\frac{r}{\alpha} \right)^2 \right]^{-\beta}$$

Note that β

The FWHM is twice the value of r at which $I(r) = 0.5 I_0$, so derivation of the value of α corresponding to a given FWHM is straightforward.

$$\begin{aligned} \left[1 + \left(\frac{r}{\alpha} \right)^2 \right]^{-\beta} &= \frac{1}{2} \\ \left[1 + \left(\frac{r}{\alpha} \right)^2 \right]^{\beta} &= 2 \\ 1 + \left(\frac{r}{\alpha} \right)^2 &= 2^{\frac{1}{\beta}} \\ \frac{r}{\alpha} &= \sqrt{2^{\frac{1}{\beta}} - 1} \\ \alpha &= \frac{\text{FWHM}}{2\sqrt{2^{\frac{1}{\beta}} - 1}} \end{aligned}$$

```
moffat <- function(r,fwhm,beta=4.765) {
  hwhm <- (fwhm/2.0)
  alpha <- hwhm*(2^(1.0/beta)-1)^(-0.5)
  p <- (1.0 + (r/alpha)^2.0)^(-1.0*beta)
  p/sum(p)
}
```

2 The PSF as derived from models of atmospheric turbulence

[Fried \(J. Opt. Soc. Am., Volume 56, p. 1372-1379\)](#) (as reported by [Trujillo et al](#)) and [Woolf \(1982\)](#) use atmospheric turbulence theory to show that the PSF is the Fourier transform of an exponential:

$$\mathcal{F}(PSF) \propto e^{-(kb)^{\frac{5}{3}}}$$

where

$$FWHM = 2.9207b$$

[Trujillo et al](#) find that this closely matches a Moffat PSF with $\beta = 4.765$. Typical values of β measured from astronomical images have lower values of β ([Saglia et al \(1993MNRAS.264..961S\)](#)), but not as low as the default value of 2.5 used by some IRAF routines.

[Saglia et al](#) propose a model PSF inspired by the form of that derived from the atmospheric model:

$$\mathcal{F}(PSF) \propto e^{-(kb)^\gamma}$$

3 Gaussian and multi-Gaussian PSFs

Gaussian PSF models are popular, and fits of multiple Gaussians can result in fits competitive with Moffat model ([Bendinelli et al 1990](#)) fits.

4 Airy Patterns and Diffraction Limited PSFs

The simplest practical diffraction limited PSF is that resulting from a uniform circular aperture, the Airy pattern. Following chapter 2 and appendix C of [Steward's Fourier Optics, an introduction](#):

$$\begin{aligned} F(u) &= \int_{-\frac{d}{2}}^{\frac{d}{2}} 2 \left(\frac{d^2}{4} - x^2 \right)^{\frac{1}{2}} e^{i2\pi ux} dx \\ &= \frac{\pi d^2}{4} \left[\frac{2J_1(\pi ud)}{\pi ud} \right] \end{aligned}$$

where J_1 is a Bessel function of the first order, first kind, a is the aperture width, and $u = \frac{\sin \theta}{\lambda} \simeq \frac{\theta}{\lambda}$.

Note that, because Airy pattern arises from Fraunhofer diffraction of an aperture of finite width, the PSF must be band limited.

To obtain an intensity from $F(u)$, it must be squared. To get the half maximum intensity,

$$\begin{aligned} \left(\frac{J_1(x)}{x} \right)^2 &= \frac{1}{2} \left(\lim_{x_0 \rightarrow 0} \frac{J_1(x_0)}{x_0} \right)^2 \\ &= \frac{1}{8} \end{aligned}$$

which happens at roughly $x \simeq 1.61633$ So we can get the FWHM:

$$\begin{aligned}\pi u d &= 1.61633 \\ \frac{\theta_{\text{HWHM}}}{\lambda} d &= \frac{1.61633}{\pi} \\ \theta_{\text{FWHM}} &= 2 \times 0.51448 \frac{\lambda}{d} = 1.029 \frac{\lambda}{d}\end{aligned}$$

This differs from the familiar expression for the diffraction limit of a telescope $\theta_0 = 1.22 \frac{\lambda}{d}$ because the latter is measured to the first trough in the diffraction pattern rather than the half maximum.

Note that θ above is in radians.

The R code to generate the Airy pattern is:

```
airy <- function(r,fwhm) {
  x <- r * 2 * 1.61633 / fwhm
  a <- 4*( besselJ(abs(x),1)/abs(x) )^2
  a[which(x==0)] <- 1.0
  a/sum(a)
}
```

5 Nyquist sampling and a band limited images

Convolution by a point spread function in image space is equivalent to multiplication by the Fourier transform of the PSF in Fourier space. PSF Fourier transforms are generally low-pass filters: they reduce the high frequency components of the image. If a PSF completely removes all frequencies higher than some μ_c , then the PSF (and any images convolved by the PSF) are band limited at μ_c .

The sampling of an image determines which Fourier frequencies can be measured from the image. If all frequencies lower than the band limit can be extracted, the image is said to be critically sampled. If the image is oversampled, then it is sampled more frequently than necessary.

If we oversample the image, we might preserve noise that otherwise would be lost, but no signal. Consider the image as a function of its Fourier transform:

$$g(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} G(u, v) e^{2\pi i \frac{ux+vy}{N}}$$

$$g(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} H(u, v) F(u, v) e^{2\pi i \frac{ux+vy}{N}} + \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} N(u, v) e^{2\pi i \frac{ux+vy}{N}}$$

By definition, if H is critically sampled,

$$H(u, v) = \Pi\left(\frac{2u}{N_c}, \frac{2v}{N_c}\right) H(u, v)$$

so

$$g(x, y) = \frac{1}{N} \sum_{u=0}^{N_c} \sum_{v=0}^{N_c} H(u, v) F(u, v) e^{2\pi i \frac{ux+vy}{N}} + \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} N(u, v) e^{2\pi i \frac{ux+vy}{N}}$$

$$g(x, y) = g_c(x, y) + \frac{1}{N} \sum_{u=N_c+1}^{N-1} \sum_{v=N_c+1}^{N-1} N(u, v) e^{2\pi i \frac{ux+vy}{N}}$$

Undersampling the image (preserving too few pixels to measure all Fourier components out to the band limit) has the same result as multiplying the fourier transform by a square function. The Fourier transform of a square function is a sinc, and a multiplication in Fourier space is a convolution in image space, so this resampling results in a convolution by a sinc function.

In practice, the true PSF is probably never critically sampled. Moffat, Gaussian, and theoretical turbulence profiles, for example, are never truly band limited; the Fourier transform of either never strictly reaches zero. However, there will be a value of N_c for which

$$H(u, v) F(u, v) \ll N(u, v)$$

In this case, including u or v greater than N_c can only reduce the signal to noise. If $g(x, y)$ is a Wiener filtered image,

$$g(x, y) = \frac{1}{N} \sum_{u=N_c+1}^{N-1} \sum_{v=N_c+1}^{N-1} G(u, v) e^{2\pi i \frac{ux+vy}{N}} \simeq 0$$

anyway.

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